■ Method

(1) Prove based on the definition.

(2) Use known set equalities or inclusions and prove through set algebra

e.g. »» **Example 3 proof:**

(1) A ∪ B=B ∪ A (Commutative Law of Union**)**

Proof: We need to prove that both $A\cup B\subseteq B\cup A$ and $B\cup A\subseteq A\cup B$ hold

$\forall x \quad x\in A\cup B$

$\Rightarrow x\in A$ **or** $x\in B$**, Then** $x\in B$ **or** $x\in A$

$\Rightarrow x\in B\cup A$

Thus, we have proven that $A\cup B\subseteq B\cup A$.

Similarly, we can prove that $B\cup A\subseteq A\cup B$.

(2) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ (Distributive Law of Union over Intersection)

Proof: We need to proof $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$ and $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$

$\forall x \quad x \in A \cup (B \cap C)$

$\Rightarrow x \in A$ *or* $(x \in B$ *and* $x \in C)$

$\Rightarrow (x \in A$ *or* $x \in B)$ *and* $(x \in A$ *or* $x \in C)$

$\Rightarrow x \in (A \cup B) \cap (A \cup C)$

Therefore $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

which proves $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$.

(3) $A \cup E = E$ (Union with the Universal Set)

Proof: According to the definition of union, we have $E \subseteq A \cup E$.

According to the definition of the universal set, we have $A \cup E \subseteq E$

(4) $A \cap E = A$ (Law of Identity)

Proof: We need to prove $A \subseteq A \cap E$ and $A \cap E \subseteq A$

By the definition of intersection, **we have $A \cap E \subseteq A$.**

For $\forall x \quad x \in A$,

By the definition of the universal set E

$x \in E$, Therefore $x \in A$ ***and*** $x \in E$,

$\Rightarrow \quad x \in A \cap E$

Thus $A \subseteq A \cap E$.

e.g. **Example 4： Prove that $A \cup (A \cap B) = A$** （Absorption Law）

Proof: Using the four identities proven in Example 3 to prove:

$A \cup (A \cap B)$

$= (A \cap E) \cup (A \cap B)$ **(**Law of Identity**)**

$= A \cap (E \cup B)$ **(**Distributive Law**)**

$= A \cap (B \cup E)$ **(**Commutative Law**)**

$= A \cap E$ **(**Law of Excluded Null**)**

$= A$ **(**Law of Identity**)**

ⓘ For the remaining basic set identities, we will not prove each one indi vidually (please prove them yourself). From now on, we will use them as known set identities.

e.g. **Example 5: Prove that $(A - B) - C = (A - C) - (B - C)$**

Proof:

$(A - C) - (B - C)$

$= (A \cap \sim C) \cap \sim(B \cap \sim C)$ (Complement Intersection Conversion)

$= (A \cap \sim C) \cap (\sim B \cup \sim\sim C)$ (De Morgan's Law))

$= (A \cap \sim C) \cap (\sim B \cup C)$ (Double Negation Law)

$= (A \cap \sim C \cap \sim B) \cup (A \cap \sim C \cap C)$ (Distributive Law)

$= (A \cap \sim C \cap \sim B) \cup (A \cap \varnothing)$ (Contradiction Law)

$= A \cap \sim C \cap \sim B$ (Zero Law, Identity Law)

$= (A \cap \sim B) \cap \sim C$ (Commutative Law, Associative Law)

$= (A - B) - C$ (Complement Intersection Conversion Law) e.g. **Example 6:** Prove $(A\cup B)\oplus(A\cup C) = (B\oplus C) - A$

Need to prove $(A\cup B)\oplus(A\cup C)$

$$=((A\cup B) - (A\cup C))\cup((A\cup C) - (A\cup B))$$

$$=((A\cup B)\cap\sim A\cap\sim C)\cup((A\cup C)\cap\sim A\cap\sim B)$$

$$=(B\cap\sim A\cap\sim C)\cup(C\cap\sim A\cap\sim B)$$

$$=((B\cap\sim C)\cup(C\cap\sim B))\cap\sim A$$

$$=((B-C)\cup(C-B))\cap\sim A$$

$$=(B\oplus C) - A$$

e.g. **Example 7：**

Let A and B be any sets, with power sets P(A) and P(B).

Prove that: ***If*** $A \subseteq B$, ***then*** $P(A) \subseteq P(B)$

Proof： $\forall x \quad x \in P(A) \Leftrightarrow x \subseteq A$

$\Rightarrow x \subseteq B$ **(Since** $A \subseteq B$**)**

$\Leftrightarrow x \in P(B)$

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e.g. **Example8:** **Proof** $A \oplus B = A \cup B - A \cap B$.

Proof $A \oplus B = (A \cap \sim B) \cup (\sim A \cap B)$

$= (A \cup \sim A) \cap (A \cup B) \cap (\sim B \cup \sim A) \cap (\sim B \cup B)$

$= (A \cup B) \cap (\sim B \cup \sim A)$

$= (A \cup B) \cap \sim (A \cap B)$

$= A \cup B - A \cap B$

1.3 Proof Methods

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- Direct Proof Method
- Indirect Proof Method
- Reductio ad Absurdum (Proof by Contradiction)
- Exhaustive Method
- Constructive Proof Method
- Vacuous Proof Method
- Trivial Proof Method
- Mathematical Induction
- Counterexample—Proof that a Proposition is False

- **Form 1:** If A, then B
- **Form 2:** A if and only if B
- **Form 3:** Prove B
- All can be reduced to **Form 1**

- **Method:** Assume A is true, prove B is true.

e.g. »» **Example 1:** If n is odd, then n^2 is also odd.

Proof:

Assume n is odd, then there exists $k \in N$,

such that $n = 2k + 1$.

Therefore,

$$n^2 = (2k + 1)^2$$
$$= 2(2k^2 + 2k) + 1$$

Thus, n^2 is odd.

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- **Indirect proof method** is a generalized proof technique that encompasses any method of proof that does not directly derive the conclusion from the premise.
- **Logical fact：** A proposition and its contrapositive are logically equivalent.
- **Method:** To prove "A → B", it is sufficient to prove "¬B → ¬A", that is, "If B is not true, then A is not true."

e.g. **Example 2:** If n^2 is odd, then n is also odd.

Proof: It suffices to prove that: If n is even, then n^2 is even. That is, prove the original proposition is true.

Assume n is even, then there exists $k \in N$, such that $n = 2k$. Therefore,

$n^2 = (2k)^2 = 2(2k^2)$

Thus, n^2 is even.

- **Proof by Contradiction** begins by assuming that the negation of the proposition to be proven is true, and then through logical reasoning, a contradiction or an impossible result is derived.
- **Method:** Let A be true, assume B is not true, and derive a contradiction.

e.g. **Example 3:** If $A-B=A$, then $A\cap B=\emptyset$.

Proof: Using proof by contradiction, assume $A\cap B\neq\emptyset$. Then there exists an element x such that

$x\in A\cap B \iff x\in A$ and $x\in B$.

Since $A-B=A$, it follows that $x\in A-B$ and $x\in B$

$\iff (x\in A$ and $x\notin B)$ and $x\in B$

$\implies x\notin B$ and $x\in B$,

which is a contradiction.

e.g. **Example 4: Prove that $\sqrt{2}$ is irrational.**

Proof: Assume $\sqrt{\mathbf{2}}$ is rational. Then there exist positive integers m and n such that $\sqrt{\mathbf{2}} = \frac{\boldsymbol{m}}{\boldsymbol{n}}$, where n≠0 and m and n have no common factors (i.e., $\frac{\boldsymbol{m}}{\boldsymbol{n}}$ is in lowest terms).

Then m=$n\sqrt{\mathbf{2}}$, squaring both sides gives $m^2=2n^2$. This implies m^2 is even, and therefore m is even. Let $m=2k$. Substituting back, we get $(2k)^2 = 2n^2$, which simplifies to $4k^2=2n^2$ or $n^2=2k^2$. This implies n^2 is even, and hence n is also even. This contradicts the assumption that $\frac{\boldsymbol{m}}{\boldsymbol{n}}$ is in lowest terms.

ⓘ **Proof by Contradiction** is a special form of indirect proof: by showing that the negation of the conclusion leads to the negation of the premise, thereby contradicting the initial assumption.

- **Definition:** Exhaustive method (also known as the method of exhaustion) is a technique used to prove a proposition by verifying all possible cases. This method is typically applied when the number of possible situations is finite and can be explicitly listed.
- **Scope of application:** It is suitable for problems where the solution space is small and manageable.
- **Proof process:** The prover needs to examine each possible case one by one and demonstrate that the proposition holds true in all these situations.
- **Characteristics:** The key to the exhaustive method lies in its completeness, ensuring that all possible cases are considered. However, it is generally impractical when dealing with a large solution space.

- **Definition: Proof by cases** is a method where the original problem is decomposed into several smaller, more manageable sub-problems, each of which is proven individually. The sum of these sub-problems covers all situations of the original problem.
- **Usage Scenario:** It is applicable when the problem inherently possesses natural classifications or when the solution can be simplified through logical division.
- **Proof Process:** The prover decomposes the problem into several non-overlapping cases based on different characteristics or conditions and proves the correctness of the proposition for each case individually.
- **Characteristics:** The focus of proof by cases lies in the effective division of the problem and the independent handling of each sub-case. This method may employ different proof strategies in different situations.

- **Definition:** The proposition to be proven is of the form = $A_1 \vee A_2 \vee \ldots \vee A_k \rightarrow B$.
- **Method:** Prove that $A_1 \rightarrow B, A_2 \rightarrow B, \ldots, A_k \rightarrow B$ are all true.

e.g. **Example 5:** Prove that max(a,max(b,c))=max(max(a,b),c).

Proof:

情况	u=max(b,c)	max(a,u)	v=max(a,b)	max(v,c)
$a \le b \le c$	c	c	b	c
$a \le c \le b$	b	b	b	b
$b \le a \le c$	c	c	a	c
$b \le c \le a$	c	a	a	a
$c \le a \le b$	b	b	b	b
$c \le b \le a$	b	a	a	a

- **Definition:** A proof by construction involves creating a specific example or object to prove the truth of a proposition. This method is typically used for "existence proofs."
- **Method:** Under the condition that A is true, construct an object with this property.

e.g. »» **Example 6:** For every positive integer n, there exist n consecutive positive composite numbers.

Proof: Let x=(n+1)!.

Then x+2,x+3,…,x+n+1 are n consecutive positive composite numbers:

For i=2,3,…,n+1, x+i is composite.

- **Constructive Proof:** A constructive proof provides one or more specific instances or examples to prove a proposition. It is applicable when demonstrating that there exist particular objects or numbers that satisfy certain conditions.
- **Non-Constructive Proof:** A non-constructive proof establishes the truth of a proposition without directly presenting specific examples. This method often relies on logical reasoning, existing theories, or theorems, or employs techniques such as proof by contradiction.
- When it is difficult to directly construct an instance that meets the required conditions, non-constructive proofs allow us to prove the existence of certain entities or the truth of certain propositions.

- A vacuous proof is commonly used to prove statements of the form "All objects satisfying a particular property P also satisfy another property Q" ($P \rightarrow Q$). This method is particularly applicable when no objects satisfy the initial property P. In such cases, the statement is considered true because there are no counterexamples to invalidate it.
- A conditional statement $P \rightarrow Q$ is false only when P is true, and Q is false. Therefore, to prove $P \rightarrow Q$ is always true using the vacuous proof method, it suffices to show that P is always false.

e.g. **Example:**

Let $n \in N$. Define P(n): If $n > 1$, then $n^2 > 1$. Prove that P(0) is true.

P(0): If $0 > 1$, then $0^2 > 1$.

Since the premise $0 > 1$ is always false, by the vacuous proof method, we can assert that P(0) is true.

- **Trivial Proof Method (**Proof by Showing the Consequent is True**)**
- The trivial proof method is used to prove propositions that are clearly true under specific conditions.
- **Method:**

  Prove that **B** is always true, without needing to assume **A** is true.

e.g. **Example:**

If a≤b, then $\mathbf{a^0 \le b^0}$.

Proof:

Based on a universally accepted mathematical fact, any number raised to the power of 0 equals 1. Therefore, regardless of the size relationship between a and b, both $\mathbf{a^0}$ and $\mathbf{a^0 \le b^0}$ equal 1. Hence, $\mathbf{a^0 \le b^0}$ holds.

ⓘ This method often appears in the base case of induction proofs.